UNIVERSITI SAINS MALAYSIA

First Semester Examination
2016/2017 Academic Session

December 2016 / January 2017

MAT 363 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. (a) Assume that *A* and *B* are two events defined on the same sample space with probabilities $P(A)=\frac{3}{4}$ and $P(B)=\frac{3}{8}$. Show that $P(A\cup B)\geq\frac{3}{4}$.

[30 marks]

(b) If the random variable *X* has the density function *f*(*x*) = 1, 0 < *x* < 1, and zero elsewhere, find the density function *g*(*y*) of the random variable $Y=-\frac{1}{\lambda}\log(1-X)$, $\lambda > 0$.

[30 marks]

(c) If *X* and *Y* are continuous random variables, show that

(i) $P(X\leq Y)=\int_{-\infty}^{\infty}F_X(y)\,f_Y(y)\,dy$, where F_X and f_Y are the marginal distribution function of *X* and the marginal density function of *Y*.

(ii) $E(X\,|\,Y=y)=E(X)$, for all values of *y*.

[40 marks]

1. *(a)* *Andaikan bahawa A dan B adalah dua peristiwa yang ditakrifkan pada ruang sampel yang sama dengan kebarangkalian* $P(A)=\frac{3}{4}$ *dan* $P(B)=\frac{3}{8}$. *Tunjukkan bahawa* $P(A\cup B)\geq\frac{3}{4}$.

[30 markah]

(b) *Jika pembolehubah rawak X mempunyai fungsi ketumpatan f(x) = 1, 0 < x < 1, dan sifar di tempat lain, cari fungsi ketumpatan g(y) untuk pembolehubah rawak* $Y=-\frac{1}{\lambda}\log(1-X)$, $\lambda > 0$.

[30 markah]

(c) *Jika X dan Y adalah pembolehubah rawak selanjar, tunjukkan bahawa*

(i) $P(X\leq Y)=\int_{-\infty}^{\infty}F_X(y)\,f_Y(y)\,dy$, *yang mana* F_X *dan* f_Y *adalah fungsi taburan sut untuk X dan fungsi ketumpatan sut untuk Y.*

(ii) $E(X\,|\,Y=y)=E(X)$, *untuk semua nilai y.*

[40 markah]

2. (a) If Z follows a standard normal distribution, use the distribution function method to show that $W = Z^2$ follows a chi square distribution with one degree of freedom.

[30 marks]

(b) Show that the function f given as $f(x) = \dfrac{\beta}{\pi\left[\beta^2 + (x-\alpha)^2\right]}$, $-\infty < x < \infty$, $\beta > 0$, is a density function.

[30 marks]

(c) If $X_1, X_2, ..., X_n$ are independent standard normal random variables, find the moment generating function (mgf) of $\dfrac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$. What is the distribution of $\dfrac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$, based on the mgf obtained?

[40 marks]

2. (a) Jika Z bertaburan normal piawai, gunakan kaedah fungsi taburan untuk menunjukkan bahawa $W = Z^2$ *bertaburan khi kuasa dua dengan satu darjah kebebasan.*

[30 markah]

(b) Tunjukkan bahawa fungsi f yang diberi sebagai $f(x) = \dfrac{\beta}{\pi\left[\beta^2 + (x-\alpha)^2\right]}$, $-\infty < x < \infty$, $\beta > 0$, *ialah suatu fungsi ketumpatan.*

[30 markah]

(c) Jika $X_1, X_2, ..., X_n$ *ialah pembolehubah rawak normal piawai yang tak bersandar, cari fungsi penjana momen (fpm) untuk* $\dfrac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$ *. Apakah taburan untuk* $\dfrac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i$ *, berdasarkan fpm yang diperoleh?*

[40 markah]

3. (a) Assume that $\{X_n\}$ is a sequence of discrete random variables with probability mass function

$$f_n(x) = \begin{cases} \dfrac{1}{n}, & \text{if } x = n \\ 1 - \dfrac{1}{n}, & \text{if } x = 0 \end{cases}.$$

Find the limiting distribution of X_n, if it exists.

[20 marks]

(b) Consider a random sample of size n from a distribution with density function $f(x;\theta) = \dfrac{2x}{\theta}$, $0 < x < \theta$; $\theta > 0$.

(i) Find the method of moments estimator of θ.

(ii) Find the maximum likelihood estimator of θ.

[40 marks]

(c) Assume that $X_1, X_2, ..., X_n$ is a random sample from the geometric distribution with density function $f(x;\theta) = \theta(1-\theta)^x I_{\{0,1,2,...\}}(x)$, $0 < \theta < 1$.

(i) Find the uniformly minimum variance unbiased estimator (UMVUE) of $\dfrac{1-\theta}{\theta}$.

(ii) Show that the variance of the UMVUE in (i) attains the Cramer-Rao's lower bound for the variance of unbiased estimators of $\dfrac{1-\theta}{\theta}$.

[40 marks]

3. (a) Andaikan bahawa $\{X_n\}$ ialah suatu jujukan pembolehubah rawak diskrit dengan fungsi jisim kebarangkalian

$$f_n(x) = \begin{cases} \dfrac{1}{n}, & \text{jika } x = n \\ 1 - \dfrac{1}{n}, & \text{jika } x = 0 \end{cases}.$$

Cari taburan penghad untuk X_n, jika ia wujud.

[20 markah]

(b) *Pertimbangkan suatu sampel rawak dengan saiz n daripada taburan dengan fungsi ketumpatan* $f(x;\theta) = \dfrac{2x}{\theta}$, $0 < x < \theta$; $\theta > 0$.

(i) *Cari penganggar kaedah momen untuk* θ.

(ii) *Cari penganggar kebolehjadian maksimum untuk* θ.

[40 markah]

(c) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *ialah suatu sampel rawak daripada taburan geometri dengan fungsi ketumpatan* $f(x;\theta) = \theta(1-\theta)^x I_{\{0,1,2,...\}}(x)$, $0 < \theta < 1$.

(i) *Cari penganggar saksama bervarians minimum secara seragam (PSVMS) untuk* $\dfrac{1-\theta}{\theta}$.

(ii) *Tunjukkan bahawa varians PSVMS dalam (i) mencapai batas bawah Cramer-Rao untuk varians penganggar-penganggar saksama bagi* $\dfrac{1-\theta}{\theta}$.

[40 markah]

4. (a) Let X be an observation from the gamma, $G\left(1, \dfrac{1}{\lambda}\right)$ distribution.

(i) Show that $\dfrac{2X}{\lambda}$ is a pivotal quantity.

(ii) Derive a 95% confidence interval for λ using the pivotal quantity in (i).

[40 marks]

(b) Assume that X has the density function $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$; $\theta > 0$. For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, a sample of size 3 is taken from the distribution. The critical region of the test is $C = \{(x_1, x_2, x_3) : x_1 x_2 x_3 \geq k\}$. Find the power function of the test.

[30 marks]

(c) Assume that $X_1, X_2, ..., X_n$ is a random sample from the normal, $N(\theta, 4)$ distribution. Find the likelihood ratio test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

[30 marks]

4. (a) *Biarkan X sebagai suatu cerapan daripada taburan gama,* $G\left(1, \frac{1}{\lambda}\right)$.

(i) *Tunjukkan bahawa* $\frac{2X}{\lambda}$ *ialah suatu kuantiti pangsian.*

(ii) *Terbitkan suatu selang keyakinan 95% untuk* λ *dengan menggunakan kuantiti pangsian dalam (i).*

[40 markah]

(b) *Andaikan bahawa X mempunyai fungsi ketumpatan* $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$; $\theta > 0$. *Untuk menguji* $H_0 : \theta \leq 1$ *lawan* $H_1 : \theta > 1$, *suatu sampel dengan saiz 3 diambil daripada taburan itu. Rantau genting untuk ujian itu ialah* $C = \{(x_1, x_2, x_3) : x_1 x_2 x_3 \geq k\}$. *Cari fungsi kuasa untuk ujian itu.*

[30 markah]

(c) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *ialah suatu sampel rawak daripada taburan normal,* $N(\theta, 4)$. *Cari ujian nisbah kebolehjadian untuk menguji* $H_0 : \theta = \theta_0$ *lawan* $H_1 : \theta \neq \theta_0$.

[30 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\, e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x} p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x\, I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$, $qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp\{\lambda(e^t-1)\}$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$, $t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x}x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha$, $t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2}x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}$, $t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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